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Bosonization and Scale Invariance on Quantum Wires

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Abstract

We develop a systematic approach to bosonization and vertex algebras on quantum wires of the form of star graphs. The related bosonic fields propagate freely in the bulk of the graph, but interact with its vertex. Our framework covers all possible interactions preserving unitarity. Special attention is devoted to the scale invariant interactions, which determine the critical properties of the system. Using the associated scattering matrices, we give a complete classification of the critical points on a star graph with any number of edges. Critical points where the system is not invariant under wire permutations are discovered. By means of an appropriate vertex algebra we perform the bosonization of fermions and solve the massless Thirring model. In this context we derive an explicit expression for the conductance and investigate its behavior at the critical points. A simple relation between the conductance and the Casimir energy density is pointed out.

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1 Introduction

Quantum graphs [1]-[11] are networks of one-dimensional wires connected at nodes. For the first time such structures have been applied some decades ago for describing the electron transport in organic molecules. More recently, quantum graphs (wires) appeared in the study of interacting one-dimensional electron gas [12]-[16]. Bosonization represents in this context a basic tool for the investigation of various phenomena like transmission through barriers, resonant multilead point-contact tunneling and conductance. Motivated by this fact, we develop in this paper a general framework for the construction of vertex operators and algebras on quantum wires, paying special attention on scale invariance. Our approach combines results from the spectral theory of the Schrödinger operator on quantum graphs [7]-[11] with an algebraic technique [17]-[23] for dealing with quantum fields with defects (impurities). This combination is quite natural because the junctions of the quantum wires can be represented as point-like defects. In this context we establish the behavior under scale transformations and classify the critical points relative to all dissipationless point-like interactions of a scalar field at the junction. Apart from isolated critical points, we discover multi-parameter families of such points which are asymmetric under permutations of single wires. Our framework applies also away from criticality, giving the possibility to analyze the renormalization group flows which interpolate between different critical points. By coupling the system to an external electric field we show that certain critical points exhibit exotic conductance properties. In some cases we detect an enhancement resulting from an analogue of Andreev's reflection [24] at the junction of the graph. In other cases the conductance is depressed with respect to the one of a single wire. Remarkably enough, these two regimes correspond to repulsive and attractive Casimir forces respectively. Finally, we investigate the effect of the four fermion interaction in the bulk solving the massless Thirring model, which is a sort of "relativistic" version of the Tomonaga-Luttinger model.

The paper is organized as follows. In section 2 we present the framework. We construct the scalar field φ and its dual $\tilde{\varphi}$ on a quantum graph and discuss the Kirchhoff rule associated with some conserved currents. The duality transformation $\varphi \leftrightarrow \tilde{\varphi}$ is also investigated. The interaction of $\{\varphi, \tilde{\varphi}\}$ at a junction of a quantum graph is studied in section 3, where the scale invariant S -matrices are classified. The vertex algebra generated by $\{\varphi, \tilde{\varphi}\}$ and the statistics of the vertex operators are described in section 4. Using the vertex algebra, we perform in section 5 the bosonization of fermions, derive the corresponding correlation functions and compute the conductance. Here we discuss also the relation between conductance and Casimir effect. In section 6 we introduce non-trivial bulk interactions and analyze

the massless Thirring model on a star graph. The impact of the bulk interaction on the conductance is also determined. Section 7 makes contact with some previous work on quantum wires and boundary conformal field theory. It contains also our conclusions and some ideas for further developments.

2 The framework

For simplicity we consider in this paper quantum wires of the form of a star graph⁴ Γ shown in Fig. 1.

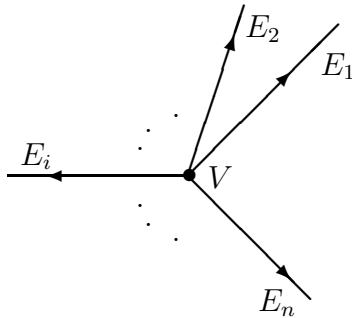


Figure 1: A star graph Γ with n edges.

Each point P in the bulk $\Gamma \setminus V$ of the graph Γ belongs to some edge E_i and can be parametrized by the pair (x, i) , where $x > 0$ is the distance of P from the vertex (junction) V along E_i . The embedding of Γ and the relative position of its edges in the ambient space are irrelevant in what follows.

2.1 The scalar field φ and its dual $\tilde{\varphi}$

The basic ingredient for bosonization is the massless scalar field φ , which satisfies

$$(\partial_t^2 - \partial_x^2) \varphi(t, x, i) = 0, \quad x > 0, \quad i = 1, \dots, n, \quad (2.1)$$

in the bulk of Γ and the vertex boundary condition

$$\sum_{j=1}^n [A_i^j \varphi(t, 0, j) + B_i^j (\partial_x \varphi)(t, 0, j)] = 0, \quad \forall t \in \mathbb{R}, \quad i = 1, \dots, n, \quad (2.2)$$

⁴Such graphs represent the building blocks for general graphs and are essential in experiments [25] as well.

A and B being in general two $n \times n$ complex matrices. Clearly the pairs $\{A, B\}$ and $\{CA, CB\}$, where C is any invertible matrix, define equivalent boundary conditions.

The results of [7] imply that the Hamiltonian of the system is self-adjoint, provided that⁵

$$A B^* - B A^* = 0, \quad (2.3)$$

and the composite matrix (A, B) has rank n . In what follows we refer to the latter as the rank condition and stress that the matrices A and B parametrize *all possible* self-adjoint extensions of the Hamiltonian from $\Gamma \setminus V$ to Γ . Requiring that φ is real (Hermitian) and imposing invariance under time reversal⁶, one infers from (2.2) that there exist an invertible matrix C , such that

$$\overline{A} = C A, \quad \overline{B} = C B, \quad (2.4)$$

where the bar stands for complex conjugation. From (2.4) and the rank condition it follows that $C \overline{C} = \overline{C} C = \mathbb{I}_n$, where \mathbb{I}_n is the $n \times n$ identity matrix. Thus, setting $C_\theta = (e^{i\theta} \mathbb{I}_n + e^{-i\theta} C)$, we conclude that $\{C_\theta A, C_\theta B\}$ are real matrices for any $\theta \in \mathbb{R}$. Since C_θ is invertible if $-e^{2i\theta}$ is not an eigenvalue of C , the boundary conditions defined by $\{A, B\}$ and $\{C_\theta A, C_\theta B\}$ are equivalent. Therefore, without loss of generality one can assume in what follows that A and B are *real*. Accordingly, (2.3) takes the form

$$A B^t - B A^t = 0, \quad (2.5)$$

where the apex t denotes transposition.

For invertible B eqs. (2.1, 2.2) define a variational problem with the action

$$I[\varphi] = I_0[\varphi] + \frac{1}{2} \sum_{i,j=1}^n \int_{-\infty}^{\infty} dt \varphi(t, 0, i) (B^{-1} A)_i^j \varphi(t, 0, j). \quad (2.6)$$

where

$$I_0[\varphi] = \frac{1}{2} \sum_{i=1}^n \int_{-\infty}^{\infty} dt \int_0^{\infty} dx [(\partial_t \varphi)(\partial_t \varphi) - (\partial_x \varphi)(\partial_x \varphi)](t, x, i). \quad (2.7)$$

The dual field $\tilde{\varphi}$ is defined by the relations

$$\partial_t \tilde{\varphi}(t, x, i) = -\partial_x \varphi(t, x, i), \quad \partial_x \tilde{\varphi}(t, x, i) = -\partial_t \varphi(t, x, i), \quad x > 0, \quad i = 1, \dots, n, \quad (2.8)$$

which imply that

$$(\partial_t^2 - \partial_x^2) \tilde{\varphi}(t, x, i) = 0, \quad x > 0, \quad i = 1, \dots, n \quad (2.9)$$

⁵We denote by $*$ the Hermitian conjugation.

⁶In other words the existence of a *antiunitary* operator T such that $T\varphi(t, x)T^{-1} = \varphi(-t, x)$.

as well.

The problem of quantizing (2.1, 2.8, 2.9) with the boundary condition (2.2) and initial conditions fixed by the equal-time canonical commutation relations

$$[\varphi(0, x_1, i_1), \varphi(0, x_2, i_2)] = [\tilde{\varphi}(0, x_1, i_1), \tilde{\varphi}(0, x_2, i_2)] = 0, \quad (2.10)$$

$$[(\partial_t \varphi)(0, x_1, i_1), \varphi(0, x_2, i_2)] = [(\partial_t \tilde{\varphi})(0, x_1, i_1), \tilde{\varphi}(0, x_2, i_2)] = -i\delta_{i_1}^{i_2} \delta(x_1 - x_2). \quad (2.11)$$

has a unique solution. It can be written in the form [26]

$$\varphi(t, x, i) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\sqrt{2|k|}} [a^{*i}(k)e^{i(|k|t-kx)} + a_i(k)e^{-i(|k|t-kx)}], \quad (2.12)$$

$$\tilde{\varphi}(t, x, i) = \int_{-\infty}^{\infty} \frac{dk \varepsilon(k)}{2\pi\sqrt{2|k|}} [a^{*i}(k)e^{i(|k|t-kx)} + a_i(k)e^{-i(|k|t-kx)}], \quad (2.13)$$

where $\varepsilon(k)$ is the sign function and $\{a_i(k), a^{*i}(k) : k \in \mathbb{R}\}$ generate the boundary (reflection-transmission) algebra [17]-[23] corresponding to the boundary condition (2.2). This is an associative algebra \mathcal{A} with identity element $\mathbf{1}$, whose generators $\{a_i(k), a^{*i}(k) : k \in \mathbb{R}\}$ satisfy the commutation relations

$$a_{i_1}(k_1) a_{i_2}(k_2) - a_{i_2}(k_2) a_{i_1}(k_1) = 0, \quad (2.14)$$

$$a^{*i_1}(k_1) a^{*i_2}(k_2) - a^{*i_2}(k_2) a^{*i_1}(k_1) = 0, \quad (2.15)$$

$$a_{i_1}(k_1) a^{*i_2}(k_2) - a^{*i_2}(k_2) a_{i_1}(k_1) = 2\pi [\delta_{i_1}^{i_2} \delta(k_1 - k_2) + S_{i_1}^{i_2}(k_1) \delta(k_1 + k_2)] \mathbf{1}, \quad (2.16)$$

and the constraints⁷

$$a_i(k) = S_i^j(k) a_j(-k), \quad a^{*i}(k) = a^{*j}(-k) S_j^i(-k). \quad (2.17)$$

The S -matrix in (2.16, 2.17) equals [7]

$$S(k) = -(A + ikB)^{-1}(A - ikB) \quad (2.18)$$

and has the following simple physical interpretation. In spite of the fact that $\{\varphi, \tilde{\varphi}\}$ propagate freely in the bulk of the graph, they interact with a specific external potential localized in the vertex and codified by the boundary condition (2.2). This interaction leads to non-trivial reflection and transmission. The associated S -matrix is given by (2.18): the diagonal element $S_i^i(k)$ is the reflection amplitude on the edge E_i , whereas $S_i^j(k)$ with $i \neq j$ is the transmission amplitude from E_i to E_j . Since the

⁷Summation over repeated upper and lower indices is understood throughout the paper.

vertex V can be viewed as a sort of impurity, it is not surprising that the algebra \mathcal{A} , appearing in the context of quantum field theory with boundaries or defects [17]–[23], represents a convenient tool for the analysis of $\{\varphi, \widetilde{\varphi}\}$ on the star graph Γ . In fact, \mathcal{A} provides a simple algebraic description of all self-adjoint extensions of the bulk Hamiltonian to the whole graph.

We assume in what follows that A and B are such that

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} S_i^j(k) = 0, \quad x > 0, \quad (2.19)$$

This condition implies the absence of bound states, ensuring that \mathcal{A} is a complete basis [26] in agreement with (2.12, 2.13). Let us mention also that (2.18) is unitary

$$S(k)^* = S(k)^{-1}, \quad (2.20)$$

and satisfies Hermitian analyticity

$$S(k)^* = S(-k). \quad (2.21)$$

Because of (2.4), one has actually that S is symmetric

$$S(k)^t = S(k), \quad (2.22)$$

which is equivalent to invariance under time reversal. Combining (2.20) and (2.21) one gets

$$S(k) S(-k) = \mathbb{I}_n, \quad (2.23)$$

which ensures the consistency of the constraints (2.17). As it should be expected, the boundary conditions associated with $\{A, B\}$ and $\{CA, CB\}$ with invertible C , lead to the same S .

2.2 Symmetries and Kirchhoff's rule

The concept of symmetry on quantum graphs needs special attention. Let $j_\nu(t, x, i)$ be a conserved current, i.e.

$$\partial_t j_t(t, x, i) - \partial_x j_x(t, x, i) = 0. \quad (2.24)$$

The time derivative of the corresponding charge is

$$\partial_t \sum_{i=1}^n \int_0^\infty dx j_t(t, x, i) = \sum_{i=1}^n \int_0^\infty dx \partial_x j_x(t, x, i) = \sum_{i=1}^n j_x(t, 0, i), \quad (2.25)$$

implying charge conservation if and only if Kirchhoff's rule

$$\sum_{i=1}^n j_x(t, 0, i) = 0 \quad (2.26)$$

holds in the vertex V . This fact is essential in our context. The invariance of the equations of motion (2.1, 2.8) under time translations implies the conservation of the current

$$\theta_{tt}(t, x, i) = \frac{1}{2} : [(\partial_t \varphi)(\partial_t \varphi) - \varphi(\partial_x^2 \varphi)] : (t, x, i) , \quad (2.27)$$

$$\theta_{tx}(t, x, i) = \frac{1}{2} : [(\partial_t \varphi)(\partial_x \varphi) - \varphi(\partial_t \partial_x \varphi)] : (t, x, i) , \quad (2.28)$$

where $: \cdots :$ denotes the normal product in the algebra \mathcal{A} . The associated Kirchhoff rule

$$\sum_{i=1}^n \theta_{tx}(t, 0, i) = 0 \quad (2.29)$$

is satisfied by construction, being a consequence [7] of (2.2), (2.5) and the rank condition on A and B . Eq. (2.29) guarantees energy conservation (no dissipation) and represents a meeting point between boundary conformal field theory [27]-[30] and the concept of scale invariance on a star graph with $n \geq 2$ edges. We will elaborate on this point in section 7, keeping for the moment the discussion as general as possible.

The equations of motion (2.1, 2.8) are also invariant under the transformations

$$\varphi(t, x, i) \longmapsto \varphi(t, x, i) + c, \quad \tilde{\varphi}(t, x, i) \longmapsto \tilde{\varphi}(t, x, i) + \tilde{c}, \quad c, \tilde{c} \in \mathbb{R}, \quad (2.30)$$

which implies the conservation of the currents

$$k_\nu(t, x, i) = \partial_\nu \varphi(t, x, i), \quad \tilde{k}_\nu(t, x, i) = \partial_\nu \tilde{\varphi}(t, x, i), \quad \nu = t, x. \quad (2.31)$$

Using the solution (2.12) and the constraints (2.17) one finds that k_ν also satisfies a Kirchhoff's rule

$$\sum_{i=1}^n k_x(t, 0, i) = 0, \quad (2.32)$$

provided that

$$\sum_{j=1}^n S_i^j(k) = 1, \quad \forall i = 1, \dots, n, \quad k \in \mathbb{R}. \quad (2.33)$$

Analogously,

$$\sum_{i=1}^n \tilde{k}_x(t, 0, i) = 0, \quad (2.34)$$

holds if

$$\sum_{j=1}^n S_i^j(k) = -1, \quad \forall i = 1, \dots, n, \quad k \in \mathbb{R}. \quad (2.35)$$

Combining (2.33) and (2.35) one draws the important conclusion that Kirchhoff's rule cannot be satisfied simultaneously for both k_ν and its dual \tilde{k}_ν . Accordingly, at most one of the charges

$$Q = \sum_{i=1}^n \int_0^\infty dx k_t(t, x, i), \quad \tilde{Q} = \sum_{i=1}^n \int_0^\infty dx \tilde{k}_t(t, x, i), \quad (2.36)$$

is t -independent, which is a first indication that the duality transformation $\varphi \leftrightarrow \tilde{\varphi}$, discussed few lines below, is not a symmetry of the boundary value problem in consideration.

For the classification of the boundary conditions at the junction, performed in the next section, it is useful to translate (2.33, 2.35) in terms of the matrices A and B entering eq. (2.2). For this purpose we introduce the vector $\mathbf{v} = (1, 1, \dots, 1)$ and observe that

$$(2.33) \iff \mathbf{v} \in \text{Ker}[S(k) - \mathbb{I}_n] = \text{Ker } A, \quad (2.37)$$

where the equality between the two kernels follows (see Lemma 3.17 of [11]) from the explicit form (2.18) of S . In simple words, Kirchhoff's rule (2.32) is satisfied if and only if the entries along each line of the matrix A sum up to 0. In a similar way one can deduce that

$$(2.35) \iff \mathbf{v} \in \text{Ker}[S(k) + \mathbb{I}_n] = \text{Ker } B. \quad (2.38)$$

We observe in conclusion that the equations of motion (2.1, 2.8) and the initial conditions (2.10, 2.11) are invariant under the exchange $\varphi \leftrightarrow \tilde{\varphi}$, but the boundary condition (2.2) breaks down the symmetry. This fact suggests to consider a similar but nonequivalent boundary value problem, defined by replacing (2.2) with

$$\sum_{j=1}^n \tilde{A}_i^j \tilde{\varphi}(t, 0, j) + \tilde{B}_i^j (\partial_x \tilde{\varphi})(t, 0, j) = 0, \quad \forall t \in \mathbb{R}, \quad i = 1, \dots, n, \quad (2.39)$$

where \tilde{A} and \tilde{B} satisfy the same conditions as A and B . The solution of the new problem is still given by (2.12, 2.13) with the substitution

$$S(k) \mapsto \tilde{S}(k) = (\tilde{A} + ik\tilde{B})^{-1}(\tilde{A} - ik\tilde{B}) \quad (2.40)$$

in the algebra \mathcal{A} . Summarizing, the duality transformation $\varphi \leftrightarrow \tilde{\varphi}$ relates the problem defined by (2.2) to the one defined by (2.39).

3 Interaction at the junction

As already observed, the fields $\{\varphi, \tilde{\varphi}\}$ freely propagate in the bulk of the star graph, but interact at its vertex. The most general interaction, preserving unitarity, is fixed by the boundary condition (2.2), where A and B satisfy the conditions given in the previous section. For each $n \geq 1$ the admissible pairs $\{A, B\}$ define a family \mathcal{S}_n of S -matrices (2.18) with intriguing structure. Before discussing the general features of \mathcal{S}_n , it is instructive to give some examples.

3.1 Examples

We start with the familiar Dirichlet

$$\varphi(t, 0, 1) = \varphi(t, 0, 2) = \dots = \varphi(t, 0, n) = 0 \quad (3.1)$$

and Newmann

$$(\partial_x \varphi)(t, 0, 1) = (\partial_x \varphi)(t, 0, 2) = \dots = (\partial_x \varphi)(t, 0, n) = 0 \quad (3.2)$$

boundary conditions, corresponding to $A = 1$, $B = 0$ and $A = 0$, $B = 1$ respectively. The associated S -matrices are $S_D = -\mathbb{I}_n$ and $S_N = \mathbb{I}_n$. For $n \geq 2$ there are many possible generalizations of the mixed (Robin) boundary condition. Let us consider for instance

$$\varphi(t, 0, 1) = \varphi(t, 0, 2) = \dots = \varphi(t, 0, n), \quad \sum_{i=1}^n (\partial_x \varphi)(t, 0, i) = \eta \varphi(t, 0, n). \quad (3.3)$$

In this case

$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & -\eta \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}, \quad (3.4)$$

leading to

$$S(k) = \frac{1}{nk + i\eta} \begin{pmatrix} (2-n)k - i\eta & 2k & 2k & \dots & 2k \\ 2k & (2-n)k - i\eta & 2k & \dots & 2k \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 2k & 2k & 2k & \dots & (2-n)k - i\eta \end{pmatrix}. \quad (3.5)$$

The boundary conditions (3.1,3.2,3.3) are symmetric under edge permutations⁸, which is clearly not the case in general. A simple asymmetric example is defined by

$$A = \frac{2}{3\rho} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \rho > 0. \quad (3.6)$$

The associated S -matrix reads

$$S(k) = \frac{1}{3(1 - \mathrm{i}\rho k)} \begin{pmatrix} -1 - 3\mathrm{i}\rho k & 2 & 2 \\ 2 & -1 & 2 - 3\mathrm{i}\rho k \\ 2 & 2 - 3\mathrm{i}\rho k & -1 \end{pmatrix}, \quad (3.7)$$

which is not invariant under the permutations $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$.

3.2 Duality and scale invariance

The basic concepts for analyzing the general structure of \mathcal{S}_n are *duality* and *scale invariance*, combined with Kirchhoff's rules (2.32,2.34). Let us describe first the duality in \mathcal{S}_n . It is easily seen that the transformation [7]

$$\{A, B\} \longmapsto \{-B, A\} \quad (3.8)$$

defines an admissible boundary condition and induces the mapping

$$S(k) \longmapsto -S(k^{-1}) \quad (3.9)$$

on \mathcal{S}_n . Therefore (3.8) relates high and low momenta, justifying the term duality transformation in \mathcal{S}_n . It is worth mentioning that if $\{A, B\}$ implies Kirchhoff's rule for the current k_ν , then its dual image $\{-B, A\}$ enforces Kirchhoff's rule for the dual current \tilde{k}_ν . We observe also that (3.8) does not preserve in general the completeness condition (2.19). In fact, it follows from (3.9) that resonant states are dual images of bound states and vice versa⁹.

Scale invariance determines the critical points and plays therefore a distinguished role. A simple example of a scale invariant S -matrix is obtained by setting $\eta = 0$ in (3.3), which leads to

$$S = \frac{1}{n} \begin{pmatrix} (2-n) & 2 & 2 & \cdots & 2 \\ 2 & (2-n) & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 2 & 2 & \cdots & (2-n) \end{pmatrix}. \quad (3.10)$$

⁸A systematic discussion of this kind of boundary conditions is given in [31].

⁹See for example (3.7)

The k -independence of (3.10) is actually a general feature of any scale invariant S -matrix and has two direct implications:

- (i) the condition (2.19) holds, implying the absence of bound states;
- (ii) the duality transformation (3.9) maps any scale invariant matrix S in $-S$ and in terms of fields is realized by the mapping $\varphi \leftrightarrow \tilde{\varphi}$.

In the scale invariant case the matrices A and B have a particular form. Since the interplay between the two terms in the boundary condition (2.2) involves a dimensional parameter, it is clear that these terms must decouple at a critical point. This is only possible if, up to a multiplication with a common invertible matrix, the non-vanishing lines of A are complementary to those of B (see e.g. (3.4) for $\eta = 0$). In other words, if A has $0 \leq p \leq n$ non-vanishing lines at certain position, B has $n-p$ such lines in the complementary position. Combining this observation with the rank condition we deduce that at any critical point $\text{rank}_A = p$ and $\text{rank}_B = n-p$. One can use therefore the parameter p for the classification of the scale invariant S -matrices. The limiting cases $p = 0$ and $p = n$ are simple: one has $A = 0$ and $B = 0$ respectively, leading to $S_N = \mathbb{I}_n$ and $S_D = -\mathbb{I}_n$. So, from now on we concentrate on the range $0 < p < n$.

3.3 Critical points for $n = 2, 3$

In what follows we focus on S -matrices satisfying Kirchhoff's rule (2.32). For $n = 2$ we have only the case $p = 1$ associated with

$$A = \begin{pmatrix} a_{11} & -a_{11} \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_{21} & b_{22} \end{pmatrix}, \quad (3.11)$$

where the entries along the lines of A sum to up to 0 as required by Kirchhoff's rule. Inserting (3.11) in (2.18) and imposing (2.5) and the rank condition one gets

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.12)$$

This S -matrix describes complete transmission without reflection and produces actually the free field on $\mathbb{R} = E_1 \cup E_2$. Relaxing Kirchhoff's rule (2.32), one has the more interesting one-parameter family of scale invariant S -matrices

$$S = \frac{1}{1 + \alpha^2} \begin{pmatrix} \alpha^2 - 1 & -2\alpha \\ -2\alpha & 1 - \alpha^2 \end{pmatrix}, \quad \alpha \in \mathbb{R}, \quad (3.13)$$

which have been studied in [23, 32].

The case $n = 3$ has a richer structure. For $p = 2$ one finds

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad (3.14)$$

which coincides with (3.10) for $n = 3$ and represents the scattering matrix for all boundary conditions given (up to a multiplication with a common invertible matrix) by

$$A = \begin{pmatrix} a_{11} & a_{12} & -a_{11} - a_{12} \\ a_{21} & a_{22} & -a_{21} - a_{22} \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad (3.15)$$

The critical point (3.14) is invariant under edge permutations and has been discovered in [14] by studying the renormalization group flow of a specific model of resonant point-contact tunnelling in a system with three wires.

In the case $p = 1$ one finds a one-parameter family of critical points

$$S = \frac{1}{1 + \alpha + \alpha^2} \begin{pmatrix} \alpha + 1 & -\alpha & \alpha(\alpha + 1) \\ -\alpha & \alpha(\alpha + 1) & \alpha + 1 \\ \alpha(\alpha + 1) & \alpha + 1 & -\alpha \end{pmatrix}, \quad \alpha \in \mathbb{R}, \quad (3.16)$$

corresponding to the class of boundary conditions

$$A = \begin{pmatrix} a_{11} & a_{12} & -a_{11} - a_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad (3.17)$$

The rank condition implies that at least one of the parameters a_{11} and a_{12} does not vanish. Renumbering the edges and rescaling A , one can take without loss of generality $a_{12} = 1$. Then $\alpha = a_{11}$. Differently from (3.14), there are no edge permutations leaving invariant (3.16) for generic α . To our knowledge the family of critical points (3.16) is new and has not been previously investigated. The peculiar behavior of the conductance in this case is described in section 5.3.

The critical points (3.12, 3.14, 3.16) are mapped by the duality transformation (3.8) to critical points, where the Kirchhoff rule (2.34) for the dual current \tilde{k}_ν holds.

3.4 Critical points for $n \geq 4$

The explicit results for $n = 3$ suggest that besides isolated critical points, the phase diagram for $n \geq 4$ involves also multi-parameter families of such points. We would

like now to determine the number of these parameters and clarify their meaning. For this purpose we first observe that the scale invariant S -matrices can be written in the form [11]

$$S = \mathbb{I}_n - 2P_{\text{Ker}B} = -\mathbb{I}_n + 2P_{\text{Ker}A}, \quad (3.18)$$

where P is the projection operator. Using that the non-vanishing lines of A are complementary to that of B , from (2.5) we deduce that in the scale invariant case

$$AB^t = 0, \quad BA^t = 0. \quad (3.19)$$

Combined with the rank condition, eqs. (3.19) imply that the lines of the matrix B form a basis in $\text{Ker}A$ and vice versa the lines of A provide a basis in $\text{Ker}B$. One has in addition that $\text{Ker}A$ and $\text{Ker}B$ are orthogonal and, assuming Kirchhoff's rule (2.32), that the vector $\mathbf{v} = (1, 1, \dots, 1)$ is orthogonal to $\text{Ker}B$. Therefore, $\text{Ker}B$ is a p -dimensional subspace embedded in $\mathbb{R}^{(n-1)}$.

Suppose we take now a $n \times n$ -matrix A with $0 < p < n$ non-vanishing lines. Our goal is to determine the number $N(n, p)$ of parameters involved in the corresponding S -matrix. According to our previous discussion, $N(n, p)$ counts the parameters needed in order to fix uniquely the position of $\text{Ker}B$ in $\mathbb{R}^{(n-1)}$. One has

$$N(n, p) = p(n - 1 - p), \quad (3.20)$$

which can be easily derived as follows. Let $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ be a basis in $\text{Ker}B$ and let the vectors $\{\mathbf{b}_1, \dots, \mathbf{b}_{n-1-p}\}$ complement it to a basis in $\mathbb{R}^{(n-1)}$. The position of $\text{Ker}B$ in $\mathbb{R}^{(n-1)}$ is determined by fixing all possible scalar products of the type $\mathbf{a}_i \cdot \mathbf{b}_j$, whose number is precisely (3.20).

Summarizing, the number of parameters characterizing any family of critical points has a simple geometric meaning related to the embedding of $\text{Ker}B$ in $\mathbb{R}^{(n-1)}$. It is also clear that a similar argument, using $\text{Ker}A$ instead of $\text{Ker}B$, gives the same result. Solving $N(n, p) = 0$, one finds $p = n - 1$ which gives the constant solution (3.10). In all other cases ($0 < p < n - 1$) the S -matrix involves $N(n, p) > 0$ real parameters. In spite of this fact, the trace of S depends exclusively on n and p . In fact, one gets from (3.18)

$$\text{Tr } S = n - 2p, \quad (3.21)$$

which will be useful in what follows.

Let us apply now (3.20) to the case $n = 4$. For $p = 3$ one finds $S = (3.10)$ with $n = 4$. In both of the remaining two cases ($p = 1, 2$) eq. (3.20) predicts a two-parameter family of critical points, which is confirmed by the explicit computation. The form of the corresponding S matrices is a bit involved and is reported in the appendix.

We observe finally that S depends on $p(n - p)$ real parameters if Kirchhoff's rule (2.32) is not imposed.

3.5 Flows

Once the critical points have been classified, one can investigate the renormalization group flows among them. The renormalization group analysis (based on instanton gas expansion and strong-weak coupling duality) performed in [14], suggests the existence of a flow between $S_N = \mathbb{I}_3$ and (3.14). Our formalism enables one to construct such a flow explicitly. Let us consider in fact the boundary conditions defined by

$$A = \rho \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \rho \geq 0. \quad (3.22)$$

The resulting S -matrix is

$$S(k) = \frac{1}{(k + i\rho)(k + 3i\rho)} \begin{pmatrix} k^2 + 2i\rho k + \rho^2 & -2\rho^2 & 2i\rho(k + i\rho) \\ -2\rho^2 & k^2 + 2i\rho k + \rho^2 & 2i\rho(k + i\rho) \\ 2i\rho(k + i\rho) & 2i\rho(k + i\rho) & k^2 - \rho^2 \end{pmatrix}, \quad (3.23)$$

which indeed interpolates between \mathbb{I}_3 ($\rho = 0$) and (3.14) ($\rho \rightarrow \infty$). From (2.6) the action along this flow is

$$I[\varphi] = I_0[\varphi] - \frac{\rho}{2} \int_{-\infty}^{\infty} dt [\varphi^2(t, 0, 1) + \varphi^2(t, 0, 2) + 2\varphi^2(t, 0, 3) - 2\varphi(t, 0, 1)\varphi(t, 0, 3) - 2\varphi(t, 0, 2)\varphi(t, 0, 3)]. \quad (3.24)$$

Another interesting flow is defined by (3.7). In the limit $\rho \rightarrow 0$ one gets the critical point (3.14), whereas for $\rho \rightarrow \infty$ one finds

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3.25)$$

which coincides with the critical point $\alpha = 0$ in the family (3.16). The flow (3.7) therefore interpolates between the isolated critical point (3.14) and the family (3.16).

Let us observe also that bound states are absent and Kirchhoff's rule (2.32) is satisfied along both the above flows. There exist therefore S -matrices which are not scale invariant but preserve (2.32).

3.6 Inverse scattering

We discuss now the reconstruction of the boundary condition (2.2) from the scattering matrix. This point is relevant because $S(k)$ is the only physical observable

in the above framework. Remarkably enough, for recovering the matrices A and B one needs only the value

$$S_0 = S(k_0), \quad (3.26)$$

of the S -matrix at arbitrary but fixed momentum $k_0 \neq 0$. According to (2.20, 2.22), we have

$$S_0^* = S_0^{-1}, \quad S_0^t = S_0. \quad (3.27)$$

Now, following [8, 10, 11] we set

$$A = \frac{1}{2}C(\mathbb{I}_n - S_0), \quad B = -\frac{i}{2k_0}C(\mathbb{I}_n + S_0), \quad (3.28)$$

where C is some invertible matrix to be fixed in a moment. One gets from (3.28)

$$AB^t = \frac{i}{2k_0}C(\mathbb{I}_n - S_0^2)C^t, \quad A + ik_0B = C, \quad (3.29)$$

which imply (2.5) and the rank condition respectively. Finally, we choose C in such a way that A and B are real. Using (3.27) we see that one can take for this purpose

$$C = e^{i\theta}\mathbb{I}_n - e^{-i\theta}S_0^*, \quad (3.30)$$

for any θ such that $e^{-2i\theta}$ is not among the eigenvalues of S_0 . Plugging (3.28) in (2.18), one gets

$$S(k) = [(k + k_0)\mathbb{I}_n + (k - k_0)S_0]^{-1}[(k - k_0)\mathbb{I}_n + (k + k_0)S_0], \quad (3.31)$$

which obviously satisfies (3.26).

Equations (3.28, 3.30) show that the boundary conditions in the vertex of the graph are determined (up to a C -factor) by the scattering data or, more precisely, by the value S_0 of the scattering matrix at fixed k_0 . This description [10] of the boundary conditions has the advantage of being non-degenerate¹⁰. Kirchhoff's rule and scale invariance are imposed in these coordinates by requiring

$$S_0\mathbf{v} = \mathbf{v}, \quad S_0^* = S_0. \quad (3.32)$$

The classification of the critical points is therefore equivalent to the classification of the S_0 -matrices satisfying (3.27) and (3.32).

¹⁰We thank the referee for this observation.

4 Vertex operators

We turn now to the construction of vertex operators on the graph Γ . Although such operators exist [23] for the general boundary condition (2.2), at a scale invariant point they have simpler and more remarkable structure. For this reason we focus here on the *scale invariant* case, introducing first the right and left chiral fields

$$\varphi_{i,R}(t-x) = \varphi(t, x, i) + \tilde{\varphi}(t, x, i), \quad \varphi_{i,L}(t+x) = \varphi(t, x, i) - \tilde{\varphi}(t, x, i). \quad (4.1)$$

Inserting (2.12,2.13) in (4.1) one gets

$$\varphi_{i,R}(\xi) = \int_0^\infty \frac{dk}{\pi\sqrt{2k}} [a^{*i}(k)e^{ik\xi} + a_i(k)e^{-ik\xi}] , \quad (4.2)$$

$$\varphi_{i,L}(\xi) = \int_0^\infty \frac{dk}{\pi\sqrt{2k}} [a^{*i}(-k)e^{ik\xi} + a_i(-k)e^{-ik\xi}] . \quad (4.3)$$

The algebraic features of the chiral fields (4.2,4.3) can be easily deduced from (2.14-2.16). One finds

$$[\varphi_{i_1,R}(\xi_1), \varphi_{i_2,R}(\xi_2)] = -i\varepsilon(\xi_{12}) \delta_{i_1}^{i_2}, \quad (4.4)$$

$$[\varphi_{i_1,L}(\xi_1), \varphi_{i_2,L}(\xi_2)] = -i\varepsilon(\xi_{12}) \delta_{i_1}^{i_2}, \quad (4.5)$$

$$[\varphi_{i_1,R}(\xi_1), \varphi_{i_2,L}(\xi_2)] = -i\varepsilon(\xi_{12}) S_{i_1}^{i_2}, \quad (4.6)$$

where $\xi_{12} \equiv \xi_1 - \xi_2$ and we have used that S is k -independent at a scale invariant point.

The chiral charges associated with (4.2,4.3) are

$$Q_{i,Z} = \frac{1}{4} \int_{-\infty}^\infty d\xi \partial_\xi \varphi_{i,Z}(\xi), \quad Z = R, L, \quad (4.7)$$

and satisfy the commutation relations

$$[Q_{i_1,R}, \varphi_{i_2,R}(\xi)] = [Q_{i_1,L}, \varphi_{i_2,L}(\xi)] = -\frac{i}{2} \delta_{i_1}^{i_2}, \quad (4.8)$$

$$[Q_{i_1,R}, \varphi_{i_2,L}(\xi)] = [Q_{i_1,L}, \varphi_{i_2,R}(\xi)] = -\frac{i}{2} S_{i_1}^{i_2}, \quad (4.9)$$

$$[Q_{i_1,Z_1}, Q_{i_2,Z_2}] = 0. \quad (4.10)$$

At this point we are ready to introduce a family of vertex operators parametrized by $\zeta = (\sigma, \tau) \in \mathbb{R}^2$ and defined by

$$v(t, x, i; \zeta) = z_i q(i; \zeta) : \exp \{ i\sqrt{\pi} [\sigma \varphi_{i,R}(t-x) + \tau \varphi_{i,L}(t+x)] \} :, \quad (4.11)$$

where the value of the normalization constant $z_i \in \mathbb{R}$ will be fixed later on and

$$q(i; \zeta) = \exp \left[i\sqrt{\pi} (\sigma Q_{i,R} - \tau Q_{i,L}) \right]. \quad (4.12)$$

The exchange properties of $v(t, x, i; \zeta)$ determine their statistics. A standard calculation shows that

$$v(t_1, x_1, i_1; \zeta_1) v(t_2, x_2, i_2; \zeta_2) = \mathcal{R}(t_{12}, x_1, i_1, x_2, i_2; \zeta_1, \zeta_2) v(t_2, x_2, i_2; \zeta_2) v(t_1, x_1, i_1; \zeta_1), \quad (4.13)$$

the exchange factor \mathcal{R} being a c-number. The statistics of $v(t, x, i; \zeta)$ is determined by the value of \mathcal{R} at space-like separation $t_{12}^2 - x_{12}^2 < 0$. By means of (4.4-4.6) and (4.8-4.10) one finds

$$\mathcal{R}(t_{12}, x_1, i_1, x_2, i_2; \zeta_1, \zeta_2) \Big|_{t_{12}^2 - x_{12}^2 < 0} = e^{-i\pi(\sigma_1\sigma_2 - \tau_1\tau_2)\varepsilon(x_{12})\delta_{i_1}^{i_2}}. \quad (4.14)$$

Therefore $v(t, x, i; \zeta)$ obey anyon (abelian braid) statistics with parameter

$$\vartheta(\zeta_1, \zeta_2) = \sigma_1\sigma_2 - \tau_1\tau_2, \quad (4.15)$$

when localized at the same wedge E_i . Otherwise, $v(t, x, i; \zeta)$ commute.

In the next section we shall bosonize two-component Dirac fermions on the graph Γ . For this purpose we take any $\zeta = (\sigma, \tau)$ with $\sigma \neq \pm\tau$ and set

$$\zeta' = (\tau, \sigma). \quad (4.16)$$

Then we define

$$V(t, x, i; \zeta) = \eta_i v(t, x, i; \zeta), \quad V(t, x, i; \zeta') = \eta'_i v(t, x, i; \zeta'), \quad (4.17)$$

where $\{\eta_i, \eta'_i\}$ are the so called Klein factors, which generate an associative algebra \mathcal{K} with identity $\mathbf{1}$ and satisfy the anticommutation relation

$$\eta_{i_1}\eta_{i_2} + \eta_{i_2}\eta_{i_1} = 2\delta_{i_1 i_2} \mathbf{1}, \quad \eta'_{i_1}\eta'_{i_2} + \eta'_{i_2}\eta'_{i_1} = 2\delta_{i_1 i_2} \mathbf{1}, \quad \eta_{i_1}\eta'_{i_2} + \eta'_{i_2}\eta_{i_1} = 0. \quad (4.18)$$

From (4.14, 4.18) one deduces that (4.17) and their Hermitian conjugates obey Fermi statistics provided that

$$\sigma^2 - \tau^2 = 2k + 1, \quad k \in \mathbb{Z}. \quad (4.19)$$

In what follows we denote by \mathcal{V}_ζ the vertex algebra generated by

$$\{V(t, x, i; \zeta), V(t, x, i; \zeta'), V^*(t, x, i; \zeta), V^*(t, x, i; \zeta')\} \quad (4.20)$$

for fixed $\zeta \in \mathbb{R}^2$. Without loss of generality one can take $\sigma > 0$, which is assumed throughout the paper. The algebra \mathcal{V}_ζ is the basic tool for bosonization.

5 Bosonization on star graphs

It is worth stressing that all the results of sections 2-4 hold on a general algebraic level and do not refer to a specific representation of the algebras \mathcal{A} and \mathcal{K} . In this sense they are universal. For the physical applications we have in mind, we fix below the Fock representation of \mathcal{A} and define a simple representation of the algebra of Klein factors \mathcal{K} .

5.1 Correlation functions

The basic correlators are

$$\langle \varphi_{i_1, Z_1}(\xi_1) \varphi_{i_2, Z_2}(\xi_2) \rangle = (\varphi_{i_1, Z_1}(\xi_1) \Omega, \varphi_{i_2, Z_2}(\xi_2) \Omega), \quad (5.1)$$

where Ω and (\cdot, \cdot) are the vacuum state and the scalar product in the Fock representation [17] of \mathcal{A} . Using the exchange relations (2.14-2.16) and the fact that $a_i(k)$ annihilate Ω one easily derives

$$\langle \varphi_{i_1, R}(\xi_1) \varphi_{i_2, R}(\xi_2) \rangle = \langle \varphi_{i_1, L}(\xi_1) \varphi_{i_2, L}(\xi_2) \rangle = \delta_{i_1}^{i_2} u(\mu \xi_{12}), \quad (5.2)$$

where

$$u(\mu \xi) = \int_0^\infty \frac{dk}{\pi} (k^{-1})_\mu e^{-ik\xi} \quad (5.3)$$

with $(k^{-1})_\mu$ defined by [18]

$$(k^{-1})_\mu = \frac{d}{dk} \ln \frac{k e^{\gamma_E}}{\mu}. \quad (5.4)$$

The derivative here is understood in the sense of distributions, γ_E is Euler's constant and $\mu > 0$ is a free parameter with dimension of mass having a well-known infrared origin. The integral (5.3), computed by means of the representation (5.4), gives

$$u(\mu \xi) = -\frac{1}{\pi} \ln(\mu |\xi|) - \frac{i}{2} \varepsilon(\xi) = -\frac{1}{\pi} \ln(i\mu \xi + \epsilon), \quad \epsilon > 0. \quad (5.5)$$

Analogously, for the mixed $L - R$ correlators one finds

$$\langle \varphi_{i_1, R}(\xi_1) \varphi_{i_2, L}(\xi_2) \rangle = \int_0^\infty \frac{dk}{\pi} (k^{-1})_\mu e^{-ik\xi_{12}} S_{i_1}^{i_2}(k), \quad (5.6)$$

$$\langle \varphi_{i_1, L}(\xi_1) \varphi_{i_2, R}(\xi_2) \rangle = \int_0^\infty \frac{dk}{\pi} (k^{-1})_\mu e^{-ik\xi_{12}} S_{i_1}^{i_2}(-k). \quad (5.7)$$

As expected, (5.6,5.7) keep track of the interaction at the junction and are quite complicated for the general S -matrix (2.18). There is however a remarkable simplification at any scale invariant point, because S is constant. One has in fact

$$\langle \varphi_{i_1,R}(\xi_1) \varphi_{i_2,L}(\xi_2) \rangle = \langle \varphi_{i_1,L}(\xi_1) \varphi_{i_2,R}(\xi_2) \rangle = S_{i_1}^{i_2} u(\mu \xi_{12}) . \quad (5.8)$$

In order to compute the correlation functions of the vertex operators (4.17), we need also a representation of the algebra \mathcal{K} of Klein factors. We adopt the one defined by the two-point correlators

$$\langle \eta_{i_1} \eta_{i_2} \rangle = \langle \eta'_{i_1} \eta'_{i_2} \rangle = \kappa_{i_1 i_2} = \begin{cases} 1, & i_1 \leq i_2, \\ -1, & i_1 > i_2, \end{cases} \quad (5.9)$$

$$\langle \eta_{i_1} \eta'_{i_2} \rangle = -\langle \eta'_{i_2} \eta_{i_1} \rangle = \kappa_{i_1 i_2} . \quad (5.10)$$

Denoting by η_i^\natural any of the factors η_i and η'_i , the n -point functions are given by

$$\langle \eta_{i_1}^\natural \cdots \eta_{i_n}^\natural \rangle = \begin{cases} 0, & n = 2k + 1, \\ \sum_{p \in \mathcal{P}_{2k}} \mathcal{E}_p \langle \eta_{p_1}^\natural \eta_{p_2}^\natural \rangle \cdots \langle \eta_{p_{2k-1}}^\natural \eta_{p_{2k}}^\natural \rangle, & n = 2k, \end{cases} \quad (5.11)$$

where the sum runs over all permutations \mathcal{P}_{2k} of the numbers $1, 2, \dots, 2k$ and \mathcal{E}_p is the parity¹¹ of the permutation p .

We turn now to the vertex correlation functions, defining first the concept of *physical* vertex operator $V_{\text{ph}}(t, x, i; \zeta)$. It is defined by selecting among all vertex correlation functions those which are invariant under the shift transformations (2.30), or equivalently, by the selection rules

$$\sum_{j=1}^n \sigma_j = \sum_{j=1}^n \tau_j = 0, \quad (5.12)$$

which are the counterpart of the “neutrality” condition in the Coulomb gas approach to conformal field theory in 1+1 dimensions. Now $V_{\text{ph}}(t, x, i; \zeta)$ are defined via their vacuum expectation values given by

$$\begin{aligned} & \langle V_{\text{ph}}(t_1, x_1, i_1; \zeta_1) \cdots V_{\text{ph}}(t_n, x_n, i_n; \zeta_n) \rangle \\ &= \begin{cases} \langle V(t_1, x_1, i_1; \zeta_1) \cdots V(t_n, x_n, i_n; \zeta_n) \rangle, & (5.12) \text{ holds,} \\ 0, & (5.12) \text{ is violated.} \end{cases} \end{aligned} \quad (5.13)$$

¹¹In other words $\mathcal{E}_p = 1$ and $\mathcal{E}_p = -1$ for even and odd permutations respectively.

Let us concentrate now on the vertex algebra \mathcal{V}_ζ . A standard computation shows that the non-trivial two-point vertex functions are

$$\begin{aligned} \langle V_{\text{ph}}(t_1, x_1, i_1; \zeta) V_{\text{ph}}^*(t_2, x_2, i_2; \zeta) \rangle = \\ z_{i_1} z_{i_2} \mu^{-[(\sigma^2 + \tau^2)\delta_{i_1}^{i_2} + 2\sigma\tau S_{i_1}^{i_2}]} \kappa_{i_1 i_2} \left[\frac{1}{i(t_{12} - x_{12}) + \epsilon} \right]^{\sigma^2 \delta_{i_1}^{i_2}} \left[\frac{1}{i(t_{12} + x_{12}) + \epsilon} \right]^{\tau^2 \delta_{i_1}^{i_2}} \\ \left[\frac{1}{i(t_{12} - \tilde{x}_{12}) + \epsilon} \right]^{\sigma\tau S_{i_1}^{i_2}} \left[\frac{1}{i(t_{12} + \tilde{x}_{12}) + \epsilon} \right]^{\sigma\tau S_{i_1}^{i_2}}, \end{aligned} \quad (5.14)$$

$$\begin{aligned} \langle V_{\text{ph}}(t_1, x_1, i_1; \zeta') V_{\text{ph}}^*(t_2, x_2, i_2; \zeta') \rangle = \\ z_{i_1} z_{i_2} \mu^{-[(\sigma^2 + \tau^2)\delta_{i_1}^{i_2} + 2\sigma\tau S_{i_1}^{i_2}]} \kappa_{i_1 i_2} \left[\frac{1}{i(t_{12} - x_{12}) + \epsilon} \right]^{\tau^2 \delta_{i_1}^{i_2}} \left[\frac{1}{i(t_{12} + x_{12}) + \epsilon} \right]^{\sigma^2 \delta_{i_1}^{i_2}} \\ \left[\frac{1}{i(t_{12} - \tilde{x}_{12}) + \epsilon} \right]^{\sigma\tau S_{i_1}^{i_2}} \left[\frac{1}{i(t_{12} + \tilde{x}_{12}) + \epsilon} \right]^{\sigma\tau S_{i_1}^{i_2}}, \end{aligned} \quad (5.15)$$

with $\tilde{x}_{12} = x_1 + x_2$. These equations suggest to take the normalization factor

$$z_i = \mu^{\frac{1}{2}(\sigma^2 + \tau^2 + 2\sigma\tau S_i^i)}. \quad (5.16)$$

In this way the vertex correlators (5.14, 5.15) are μ -independent, when localized on the same edge.

Performing in (5.14) the scaling transformation $t_i \mapsto \varrho t_i$ and $x_i \mapsto \varrho x_i$ one obtains

$$\langle V_{\text{ph}}(\varrho t_1, \varrho x_1, i_1; \zeta) V_{\text{ph}}^*(\varrho t_2, \varrho x_2, i_2; \zeta) \rangle = \varrho^{-D_{i_1}^{i_2}} \langle V_{\text{ph}}(t_1, x_1, i_1; \zeta) V_{\text{ph}}^*(t_2, x_2, i_2; \zeta) \rangle, \quad (5.17)$$

where

$$D = (\sigma^2 + \tau^2)\mathbb{I}_n + 2\sigma\tau S. \quad (5.18)$$

The scaling dimensions d_i are determined by the eigenvalues of the matrix D . Diagonalizing (5.18) one finds

$$d_i = \frac{1}{2}(\sigma^2 + \tau^2) + \sigma\tau s_i, \quad i = 1, \dots, n, \quad (5.19)$$

where s_i are the eigenvalues of S . From (2.23) one infers that in the scale invariant case $s_i = \pm 1$, which implies

$$d_i = \frac{1}{2}(\sigma + s_i \tau)^2 \geq 0. \quad (5.20)$$

Recalling that the same vertex operator on the line \mathbb{R} has dimension

$$d_{\text{line}} = \frac{1}{2}(\sigma^2 + \tau^2), \quad (5.21)$$

we see that the interaction at the junction affects the scaling dimensions. More precisely, the deviation of d_i from d_{line} reflects the interaction between the left and right chiral fields at the junction.

5.2 Bosonization

The massless Dirac equation on the star graph Γ is

$$(\gamma_t \partial_t - \gamma_x \partial_x) \psi(t, x, i) = 0, \quad (5.22)$$

where

$$\psi(t, x, i) = \begin{pmatrix} \psi_1(t, x, i) \\ \psi_2(t, x, i) \end{pmatrix}, \quad \gamma_t = \gamma^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_x = -\gamma^x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (5.23)$$

The standard vector and axial currents are

$$j_\nu(t, x, i) = \bar{\psi}(t, x, i) \gamma_\nu \psi(t, x, i), \quad j_\nu^5(t, x, i) = \bar{\psi}(t, x, i) \gamma_\nu \gamma^5 \psi(t, x, i), \quad (5.24)$$

with $\bar{\psi} \equiv \psi^* \gamma_t$ and $\gamma^5 \equiv -\gamma_t \gamma_x$. From eq.(5.22) it follows that both j_ν and j_ν^5 are conserved. Moreover, the γ^5 -identities $\gamma_t \gamma^5 = -\gamma_x$ and $\gamma_x \gamma^5 = -\gamma_t$ imply the relations

$$j_t^5 = -j_x, \quad j_x^5 = -j_t. \quad (5.25)$$

Our goal now is to quantize (5.22) using the vertex algebra \mathcal{V}_ζ and to express the currents in terms of $\{\varphi, \tilde{\varphi}\}$. For this purpose we set $\zeta = (\sigma > 0, 0)$ and define

$$\psi_1(t, x, i) = \frac{1}{\sqrt{2\pi}} V_{\text{ph}}(t, x, i; \zeta), \quad \psi_2(t, x, i) = \frac{1}{\sqrt{2\pi}} V_{\text{ph}}(t, x, i; \zeta'). \quad (5.26)$$

One easily verifies that (5.26) satisfy the Dirac equation (5.22) and obey Fermi statistics if

$$\sigma^2 = 2k + 1, \quad k \in \mathbb{N}, \quad (5.27)$$

For the equal-time anticommutators of ψ_a and ψ^{*a} ($a = 1, 2$) one finds

$$\{\psi_{a_1}(0, x_1, i_1), \psi^{*a_2}(0, x_2, i_2)\} = \frac{(-1)^k}{(2k)!} \delta^{(2k)}(x_{12}) \delta_{i_2}^{i_1} \delta_{a_1}^{a_2}, \quad (5.28)$$

showing that the conventional canonical fermions are obtained for $k = 0$. We analyze below the general case $k \in \mathbb{N}$.

The next step is to construct the quantum currents (5.24). We adopt the point-splitting procedure, considering the limit

$$j_\nu(t, x, i) = \frac{1}{2} \lim_{\epsilon \rightarrow +0} Z(\epsilon) \left[\bar{\psi}(t, x, i) \gamma_\nu \psi(t, x + \epsilon, i) + \bar{\psi}(t, x + \epsilon, i) \gamma_\nu \psi(t, x, i) \right], \quad (5.29)$$

where $Z(\epsilon)$ implements the renormalization. The basic general formula for evaluating (5.29) is obtained by normal ordering the product $V_{\text{ph}}^*(t, x + \epsilon, i; \zeta) V_{\text{ph}}(t, x, i; \zeta)$. After some algebra [23], one derives the renormalization constant

$$Z(\epsilon) = \frac{-\pi \epsilon^{\sigma^2 - 1}}{\sigma \sin\left(\frac{\pi}{2} \sigma^2\right)}, \quad (5.30)$$

which leads, performing the limit in (5.29), to the conserved current

$$j_\nu(t, x, i) = \sqrt{\pi} \partial_\nu \varphi(t, x, i) = \sqrt{\pi} k_\nu(t, x, i). \quad (5.31)$$

Thus one recovers on the star graph Γ the same type of relation as in conventional bosonization [33].

In analogy with (5.29) we introduce the axial current by

$$j_\nu^5(t, x, i) = \frac{1}{2} \lim_{\epsilon \rightarrow +0} Z(\epsilon) \left[\bar{\psi}(t, x, i) \gamma_\nu \gamma^5 \psi(t, x + \epsilon, i) + \bar{\psi}(t, x + \epsilon, i) \gamma_\nu \gamma^5 \psi(t, x, i) \right], \quad (5.32)$$

The vector current result and the γ^5 -identities directly imply that the limit in the right hand side of (5.32) exists and

$$j_\nu^5(t, x, i) = \sqrt{\pi} \partial_\nu \tilde{\varphi}(t, x, i) = \sqrt{\pi} \tilde{k}_\nu(t, x, i). \quad (5.33)$$

Eq. (2.8) shows that the relations (5.25) are respected after bosonization as well.

Another essential aspect is the Kirchhoff rule. The results of section 2.1 imply that the currents j_ν and j_ν^5 cannot satisfy simultaneously Kirchhoff's rule. In what follows we assume (2.33), which guaranties the conservation of the vector charge. The case when j_ν^5 satisfies Kirchhoff's rule can be analyzed along the same lines.

Eqs.(5.31,5.33) imply that the boundary conditions on ψ at the vertex of the star graph are most conveniently formulated in terms of the currents, which are the simplest observables of the fermion field. Combining (2.2) with (5.31,5.33) one obtains

$$\sum_{j=1}^n A_i^j \int_{+0}^{\infty} dx j_x(t, x, j) = \sum_{j=1}^n B_i^j j_x(t, 0, j). \quad (5.34)$$

where we have used that $\lim_{x \rightarrow \infty} \varphi(t, x, i) = 0$ holds on the subspace generated by the currents j_ν and j_ν^5 .

5.3 Conductance

In order to explore the conductance properties of a quantum wire with the form of a star graph, we study here the linear response of the current j_ν to a classical external potential A_ν minimally coupled to ψ , namely

$$\gamma^\nu[\partial_\nu + iA_\nu(t, x, i)]\psi(t, x, i) = 0. \quad (5.35)$$

The main step is to extend the bosonization procedure of the previous section, deriving an action in terms of φ and A_ν which implements the dynamics defined by (5.35) and preserves its invariance under the local gauge transformations

$$\psi(t, x, i) \mapsto e^{i\Lambda(t, x, i)}\psi(t, x, i), \quad (5.36)$$

$$A_\nu(t, x, i) \mapsto A_\nu(t, x, i) - \partial_\nu \Lambda(t, x, i). \quad (5.37)$$

For this purpose we first observe that according to (5.26) the transformation (5.36) is implemented by the shift

$$\varphi(t, x, i) \mapsto \varphi(t, x, i) + \frac{1}{\sigma\sqrt{\pi}}\Lambda(t, x, i), \quad \tilde{\varphi}(t, x, i) \mapsto \tilde{\varphi}(t, x, i), \quad (5.38)$$

where σ satisfies (5.27). Therefore, switching on A_ν the bosonized current becomes by gauge invariance

$$j_\nu(t, x, i) = \sqrt{\pi}\partial_\nu\varphi(t, x, i) + \frac{1}{\sigma}A_\nu(t, x, i), \quad (5.39)$$

The action, which ensures the conservation of (5.39), is

$$I[\varphi, A_\nu] = \frac{1}{2} \sum_{i=1}^n \int_{-\infty}^{\infty} dt \int_0^{\infty} dx \left[\partial^\nu \varphi \partial_\nu \varphi + \frac{2}{\sigma\sqrt{\pi}} \partial^\nu \varphi A_\nu + \frac{1}{\sigma^2 \pi} A^\nu A_\nu \right] (t, x, i). \quad (5.40)$$

In fact, varying (5.40) with respect to φ , one obtains

$$\partial^\nu \partial_\nu \varphi(t, x, i) + \frac{1}{\sigma\sqrt{\pi}} \partial^\nu A_\nu(t, x, i) = 0, \quad (5.41)$$

which is precisely the conservation of (5.39). The interaction Hamiltonian associated to (5.40) is

$$H_{\text{int}}(t) = \sum_{i=1}^n \int_0^{\infty} dx \left[\frac{1}{\sigma\sqrt{\pi}} \partial_x \varphi A_x - \frac{1}{2\sigma^2 \pi} A^\nu A_\nu \right] (t, x, i). \quad (5.42)$$

At this stage we are ready to derive the expectation value $\langle j_x(t, x, i) \rangle_{A_\nu}$ in the external field A_ν . Keeping in mind that A_ν is classical, the linear response theory [34] gives

$$\begin{aligned} \langle j_x(t, x, i) \rangle_{A_\nu} &= \langle j_x(t, x, i) \rangle + i \int_{-\infty}^t d\tau \langle [H_{\text{int}}(\tau), j_x(t, x, i)] \rangle = \\ &= \frac{1}{\sigma} A_x(t, x, i) + \frac{i}{\sigma} \sum_{j=1}^n \int_{-\infty}^t d\tau \int_0^\infty dy A_y(\tau, y, j) \langle [\partial_y \varphi(\tau, y, j), \partial_x \varphi(t, x, i)] \rangle. \end{aligned} \quad (5.43)$$

Let us consider now a uniform electric field $E(t, i) = \partial_t A_x(t, i)$ in the Weyl gauge $A_t = 0$. Using (4.4-4.6) one derives from (5.43)

$$\langle j_x(t, 0, i) \rangle_{A_x} = \frac{1}{2\sigma} \sum_{j=1}^n (\delta_i^j - S_i^j) A_x(t, j), \quad (5.44)$$

which leads to the conductance tensor

$$G_i^j = \frac{1}{2\sigma} (\delta_i^j - S_i^j). \quad (5.45)$$

Because of (2.33) G_i^j satisfies Kirchhoff's rule

$$\sum_{j=1}^n G_i^j = 0, \quad i = 1, \dots, n, \quad (5.46)$$

representing an useful check of the whole derivation.

By means of the same technique and conventions one obtains for a single infinite wire (without any junction)

$$G_{\text{line}} = \frac{1}{2\sigma}, \quad (5.47)$$

which allows one to rewrite (5.45) in the form

$$G_i^j = G_{\text{line}} (\delta_i^j - S_i^j). \quad (5.48)$$

The unitarity of the S -matrix implies $|S_i^i| \leq 1$, leading to the simple bound

$$0 \leq G_i^i \leq 2G_{\text{line}}, \quad (5.49)$$

where we have used that σ and therefore G_{line} are positive. Another constraint on the diagonal elements of G is obtained from (3.21), which gives the sum rule

$$\text{Tr } G = 2p G_{\text{line}}, \quad (5.50)$$

where p is the rank of the matrix A entering the boundary condition (2.2).

It is instructive at this point to consider some examples. For the critical points (3.10) one has for instance (no summation over i)

$$G_i^i = 2 \left(1 - \frac{1}{n} \right) G_{\text{line}} \xrightarrow{n \rightarrow \infty} 2G_{\text{line}} , \quad (5.51)$$

showing an enhancement with respect to G_{line} for $n \geq 3$. This remarkable phenomenon has been discovered in [14], where it has been interpreted as a result of the so called Andreev reflection [24]. Notice also that the enhancement is growing with the number of wires n .

A closer examination of the case $n = 3$ reveals another interesting feature. At the critical point (3.14) all three edges have the same enhanced conductance

$$G_1^1 = G_2^2 = G_3^3 = \frac{4}{3} G_{\text{line}} . \quad (5.52)$$

The situation is quite different however for the family of critical points (3.16). In that case the sum rule (5.50) gives

$$G_1^1(\alpha) + G_2^2(\alpha) + G_3^3(\alpha) = 2G_{\text{line}} , \quad \alpha \in \mathbb{R} . \quad (5.53)$$

We have plotted the conductance of all three edges in Fig. 2. Besides of the domains

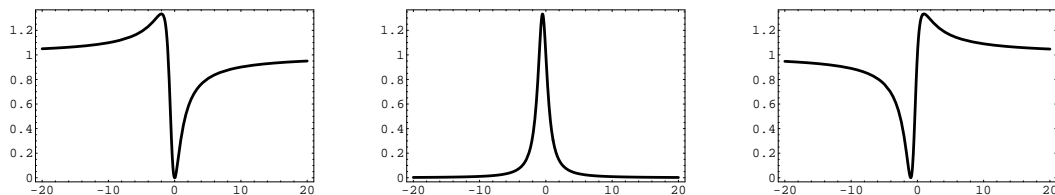


Figure 2: Plots of $G_1^1(\alpha)$ (left), $G_2^2(\alpha)$ (center) and $G_3^3(\alpha)$ (right) for $G_{\text{line}} = 1$.

of enhancement (the maxima of the curves are exactly at $4G_{\text{line}}/3$), there are domains where the conductance is depressed and points (the minima), where it actually vanishes. This characteristic property of the new critical points, discovered in the present paper, may have interesting applications. Let us observe finally that for $n = 3$ one has by inspection

$$0 \leq G_i^i|_{n=3} \leq \frac{4}{3} G_{\text{line}} , \quad (5.54)$$

which is sharper than the unitarity bound (5.49).

It is easy to see that the conductance for the axial current j_ν^5 is obtained from (5.48) by the substitution $S_i^j \mapsto -S_i^j$.

Eq. (5.48) represents a nice and universal formula which holds at any critical point and determines the conductance in terms of the interaction at the junction. An analogous expression has been derived in [26] for another physical observable - the Casimir energy density $\mathcal{E}_C(x, i)$. One has

$$\mathcal{E}_C(x, i) = -\frac{1}{8\pi x^2} S_i^i, \quad (5.55)$$

establishing a direct relation between conductance and Casimir effect on quantum wires. Eq. (5.55) implies that in the case of enhanced conductance ($S_i^i < 0$), the energy density $\mathcal{E}_C(x, i)$ gives rise to a repulsive Casimir force.¹² If instead the conductance is depressed, the Casimir force is attractive.

We would like to mention in conclusion that the above technique applies to the derivation of the conductance away from the critical points as well. Adopting the general correlators (5.2, 5.6, 5.7) one obtains

$$G_i^j(\omega) = G_{\text{line}} [\delta_i^j - S_i^j(\omega)], \quad (5.56)$$

where ω is the frequency of the Fourier transform $\hat{A}_x(\omega, i)$ of the external field $A_x(t, i)$ applied to the system.

In the next section we shall introduce non-trivial scale invariant bulk interactions and investigate their impact on the conductance.

6 The massless Thirring model

The classical dynamics of the massless Thirring model [37] is governed by the equation of motion

$$i(\gamma_t \partial_t - \gamma_x \partial_x) \Psi(t, x, i) = \lambda [\gamma_t J_t(t, x, i) - \gamma_x J_x(t, x, i)] \Psi(t, x, i), \quad x \neq 0, \quad (6.1)$$

where $\lambda > 0$ is the coupling constant and J_ν is the conserved current

$$J_\nu(t, x, i) = \bar{\Psi}(t, x, i) \gamma_\nu \Psi(t, x, i). \quad (6.2)$$

The system is scale invariant and can be quantized by means of the vertex algebra \mathcal{V}_ζ with $\zeta = (\sigma > 0, \tau)$. We set

$$\Psi_1(t, x, i) = \frac{1}{\sqrt{2\pi}} V_{\text{ph}}(t, x, i; \zeta), \quad \Psi_2(t, x, i) = \frac{1}{\sqrt{2\pi}} V_{\text{ph}}(t, x; \zeta'), \quad (6.3)$$

¹²There is recently a growing interest [35, 36] in this phenomenon.

where σ and τ satisfy (4.19) in order to have Fermi statistics.

The quantum current J_ν is constructed in analogy with (5.29), setting

$$J_\nu(t, x, i) = \frac{1}{2} \lim_{\epsilon \rightarrow +0} Z(x, i; \epsilon) \left[\bar{\Psi}(t, x, i) \gamma_\nu \Psi(t, x + \epsilon, i) + \bar{\Psi}(t, x + \epsilon, i) \gamma_\nu \Psi(t, x, i) \right]. \quad (6.4)$$

Using the short distance expansion of the product $V_{\text{ph}}^*(t, x + \epsilon, i; \zeta) V_{\text{ph}}(t, x, i; \zeta)$ one finds

$$Z(x, i; \epsilon) = \frac{-\pi \epsilon^{\sigma^2 + \tau^2 - 1} (2x)^{\sigma \tau S_i^i}}{(\sigma - \tau) \sin \left[\frac{\pi}{2} (\sigma^2 - \tau^2) \right]}, \quad (6.5)$$

and

$$J_\nu(t, x) = \sqrt{\pi} \partial_\nu \varphi(t, x). \quad (6.6)$$

The x and i -dependence of Z are not surprising because translations along the edges and their permutations are not symmetries in general. Because of (6.6) the quantum equation of motion takes the form

$$i(\gamma_t \partial_t - \gamma_x \partial_x) \Psi(t, x, i) = \lambda \sqrt{\pi} : (\gamma_t \partial_t \varphi - \gamma_x \partial_x \varphi) \Psi : (t, x, i). \quad (6.7)$$

Now, using the explicit form (6.3) of Ψ , one easily verifies that (6.7) is satisfied provided that

$$\tau = -\frac{1}{2} \lambda. \quad (6.8)$$

Combining eq. (4.19) and eq. (6.8), we obtain for σ

$$\sigma = \sqrt{\frac{\lambda^2}{4} + (2k + 1)}, \quad (6.9)$$

where

$$k \in \mathbb{Z}, \quad k \geq -\frac{\lambda^2 + 4}{8}, \quad (6.10)$$

ensuring that $\sigma \in \mathbb{R}$. The freedom associated with k is present also in the Thirring model on the line. Since Lorentz invariance is preserved there, it is natural to require that the Lorentz spin of Ψ takes the canonical value $1/2$, which fixes $k = 0$.

For deriving the conductance of the Thirring model one can apply the result of section 5.3. One introduces an external field A_ν minimally coupled to Ψ and observes that the local gauge transformations of Ψ are now implemented by the shift

$$\varphi(t, x, i) \mapsto \varphi(t, x, i) + \frac{1}{(\sigma + \tau) \sqrt{\pi}} \Lambda(t, x, i), \quad \tilde{\varphi}(t, x, i) \mapsto \tilde{\varphi}(t, x, i). \quad (6.11)$$

Performing the same steps as in section 5.3, one gets

$$G_i^j = \frac{1}{2(\sigma + \tau)} (\delta_i^j - S_i^j) , \quad (6.12)$$

where the bulk interaction is captured by the factor

$$\frac{1}{2(\sigma + \tau)} = \frac{1}{\sqrt{\lambda^2 + 4(2k + 1)} - \lambda} . \quad (6.13)$$

As expected, eqs. (6.12, 6.13) reproduce (5.45) for $\lambda = 0$. We see also that the effect of the current-current bulk interaction on the conductance is an overall λ -dependent renormalization of (5.45). Since the bulk interaction preserves scale invariance, all critical points of the Thirring model on a star graph are those described in section 3. From (5.20) the scaling dimensions at a given critical point are

$$d_i = \frac{1}{4} \left[\sqrt{\lambda^2 + 4(2k + 1)} - s_i \lambda \right]^2 , \quad (6.14)$$

where s_i are the eigenvalues of the related S -matrix.

7 Remarks and conclusions

The interest in various aspects of quantum field theory and critical phenomena on star graphs is not new and is growing recently. Among others, we already mentioned the very inspiring papers [12]-[16] on this subject. The interaction at the junction of the wires is implemented there by means of a Lagrangian $\mathcal{L}_{\text{int}}(t, 0, i)$, localized at the vertex $x = 0$ of the star graph. After bosonization, $\mathcal{L}_{\text{int}}(t, 0, i)$ involves typically exponential interactions of the bulk scalar fields φ and $\tilde{\varphi}$. The models arising this way are investigated by various methods including conventional perturbation theory, perturbative conformal field theory, instanton gas expansion, functional renormalization group and so on. In this context some critical points and the interpolating renormalization group flows have been determined.

In the present work we propose an alternative strategy, based on the point-like character of the vertex interactions, the powerful theory of self-adjoint extension of Hermitian operators on a graph [7]-[11] and the algebraic technique [17]-[23] for dealing with defects. Within the new approach we establish the complete classification of the critical points of the massless scalar field φ on a star graph with any number of edges. The main idea is to study the bulk dynamics of φ , isolating first the vertex of the graph. The vertex and the associated interaction are recovered

afterwards constructing all possible unitarity preserving extensions of the bulk theory to the whole graph. We derived (sect. 3) in this way new families of critical points with peculiar conductance properties (sect. 5). Besides domains of enhancement, we discover also critical points in which the conductance is depressed. It turns out that the Casimir force has a different sign in these two regimes: enhancement of the conductance corresponds to a repulsion and depression to attraction. It is worth stressing that our framework applies also away from criticality. Indeed, we were able to construct explicitly the renormalization group flows for a broad class of boundary conditions and associate with them a simple action $I[\varphi]$ which, instead of being exponential, is quadratic in φ . Non-trivial bulk interactions have been treated by this technique as well. We focused (sect. 6) on the Thirring model, which is a “relativistic” generalization of the Tomonaga-Luttinger model.

The relation of our framework to boundary conformal field theory (BCFT) [27]–[30] is another interesting issue. The key point for understanding the interplay between BCFT and scale invariance on quantum graphs is the Kirchhoff rule (2.29) for the energy-momentum tensor θ_{tx} . For $n = 1$ this rule implies the absence of momentum flow across the boundary and is precisely the starting point of conventional BCFT. In fact, for $n = 1$ our approach reproduces the results of BCFT for the scalar field φ . The relative scale-invariant boundary conditions are the Dirichlet and Neumann conditions, which correspond to *complete reflection* from the boundary. A new phenomenon takes place for $n \geq 2$. In fact, besides reflection one can have in this case also *non-trivial transmission* between the different edges. In other words, $\theta_{tx}(t, 0, i)$ need not to vanish separately for any i , still respecting (2.29). The spectrum of scale-invariant boundary conditions for $n \geq 2$ is richer (sect 3), the scale dimensions are affected by the interaction at the junction (sect. 5) and one is naturally led to a generalization of the $n = 1$ BCFT. Besides the isolated critical points, this generalization involves multi-parameter families of such points, which shed new light on the critical properties of quantum wires. For this reason we believe that conformal field theory, defined on star graphs by Kirchhoff’s rule (2.29) independently of a particular set of fundamental fields (Lagrangians), needs further attention.

It will be interesting to explore also the concept of integrability on quantum graphs. A first step in this direction is the analysis [26] of the nonlinear Schrödinger equation on a star graph. The extension of the above framework to finite temperature and/or generic quantum graphs represents also a challenging open problem, whose solution will surely help for better understanding the physics of quantum wires.

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Appendix

As explained in section 3.4, there exist two families of $n = 4$ critical points, each one depending on two parameters $\alpha_1, \alpha_2 \in \mathbb{R}$. For $p = 1$ the S -matrix is defined by:

$$\begin{aligned} S_1^1 &= \frac{1}{\Delta_1}(\alpha_1 + \alpha_1^2 + \alpha_2 + \alpha_1\alpha_2 + \alpha_2^2), \\ S_2^2 &= \frac{1}{\Delta_1}(1 + \alpha_1 + \alpha_1^2 + \alpha_2 + \alpha_1\alpha_2), \\ S_3^3 &= \frac{1}{\Delta_1}(1 + \alpha_1 + \alpha_2 + \alpha_1\alpha_2 + \alpha_2^2), \\ S_4^4 &= -\frac{1}{\Delta_1}(\alpha_1 + \alpha_2 + \alpha_1\alpha_2), \end{aligned}$$

$$\begin{aligned} S_1^2 &= -\frac{1}{\Delta_1}\alpha_2, & S_1^3 &= -\frac{1}{\Delta_1}\alpha_1, & S_1^4 &= \frac{1}{\Delta_1}(1 + \alpha_1 + \alpha_2), \\ S_2^3 &= -\frac{1}{\Delta_1}\alpha_1\alpha_2, & S_2^4 &= \frac{1}{\Delta_1}\alpha_2(1 + \alpha_1 + \alpha_2), & S_3^4 &= \frac{1}{\Delta_1}\alpha_1(1 + \alpha_1 + \alpha_2), \end{aligned}$$

with $\Delta_1 = 1 + \alpha_1 + \alpha_1^2 + \alpha_2 + \alpha_1\alpha_2 + \alpha_2^2$. The remaining entries are recovered by symmetry.

Analogously, the S -matrix of the family corresponding to $p = 2$ is given by:

$$\begin{aligned} S_1^1 &= \frac{1}{\Delta_2}[3\alpha_1^2 + 2\alpha_1(1 - \alpha_2) - (1 + \alpha_2)^2], \\ S_2^2 &= \frac{1}{\Delta_2}[-1 - \alpha_1^2 + 2\alpha_2 + 3\alpha_2^2 - 2\alpha_1(1 + \alpha_2)], \\ S_3^3 &= \frac{1}{\Delta_2}[3 - \alpha_1^2 + 2\alpha_2 - \alpha_2^2 + 2\alpha_1(1 + \alpha_2)], \\ S_4^4 &= -\frac{1}{\Delta_2}[\alpha_1^2 + 2\alpha_1(1 - \alpha_2) + (1 + \alpha_2)^2], \end{aligned}$$

$$\begin{aligned}
S_1^2 &= \frac{2}{\Delta_2}(1 + \alpha_1 + \alpha_2 + 2\alpha_1\alpha_2), & S_1^3 &= \frac{2}{\Delta_2}[\alpha_2(1 + \alpha_2) - \alpha_1(2 + \alpha_2)], \\
S_1^4 &= \frac{2}{\Delta_2}(1 + \alpha_1 - \alpha_1\alpha_2 + \alpha_2^2), & S_2^3 &= \frac{2}{\Delta_2}(\alpha_1 + \alpha_1^2 - 2\alpha_2 - \alpha_1\alpha_2), \\
S_2^4 &= \frac{2}{\Delta_2}(1 + \alpha_1^2 + \alpha_2 - \alpha_1\alpha_2), & S_3^4 &= \frac{2}{\Delta_2}(\alpha_1 + \alpha_1^2 + \alpha_2 + \alpha_2^2),
\end{aligned}$$

where $\Delta_2 = 3 + 3\alpha_1^2 + 2\alpha_1(1 - \alpha_2) + 2\alpha_2 + 3\alpha_2^2$.

As a check on the above S -matrices one can verify the validity of (2.33) and (3.21).

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